**MATH 453 :** Numerical Methods III **(4-0-4)** 11/19/19

**Catalog Description:** Numerical solution of partial differential equations. Numerical solution of boundary value problems and initial-value problems using finite difference and other methods. Analysis of stability, accuracy, and implementations of methods.

**Course Objectives:** After completing this course, students will be able to

1. Derive and implement finite difference stencils to numerically approximate solutions of elliptic, parabolic, and hyperbolic partial differential equations in one and two dimensions.
2. Compute numerical solution approximations and compare with exact (known) solutions.
3. Compute convergence rates and computationally analyze the error of numerical approximation methods.

**Learning Outcomes and Performance Criteria**

1. Demonstrate an understanding of the terminology relevant to numerical solutions of partial differential equations.

   Core Criteria:

   (a) Define and explain the difference between and ordinary and partial differential equations.

   (b) Show how to classify a partial differential equation as parabolic, elliptic, or hyperbolic and explain what these terms mean and how these classifications affect the solutions of such equations.

   (c) Define at least one (physical) example of each of the three classes of partial differential equations in one, two, and three dimensions.

   (d) Explain and give at least one example of the terms: Initial-value problem, Boundary value problem, Neumann and Dirichlet boundary conditions for both ordinary and partial differential equations.

   (e) Characterize the propagation of numerically induced error. Identify "numerical diffusion" and "numerical dispersion", for instance numerical smoothing of a non-smooth waveform.

   (f) Derive, explain, and demonstrate failure to account for the CFL (Courant-Friedrichs-Lewy) condition in both one and two dimensions.

2. Use Taylor series to create numerical routines (stencils) to approximate the solutions of partial differential equations in one dimension.

   Core Criteria:

   (a) Derive and implement finite difference stencil to approximate the solution to a one-dimensional, initial-value, boundary-value, heat equation over a given spacial domain for a given time interval with the following methods:

      Forward-time centered-space (explicit).

      Backward-time centered-space (implicit).

      The Crank-Nicolson method.

(b) Derive and implement the method of (explicit) centered finite differences to create a series of snapshots of the solution of the following vibrating string problem: $u_{tt} = c^2 u_{xx}$ for $0 \leq x \leq a$, with $t = [0, T]$. The boundary and initial conditions are $u(x, 0) = f(x), \quad u_t(x, 0) = F(x), \quad u(0, t) = u(a, t) = 0$.

Additional Criteria:

(a) Derive and implement the method of finite volumes to create a series of snapshots of the solution of the following vibrating string problem: $u_{tt} = c^2 u_{xx}$ for $0 \leq x \leq a$, with $t = [0, T]$. The boundary and initial conditions are $u(x, 0) = f(x), \quad u_t(x, 0) = F(x), \quad u(0, t) = u(a, t) = 0$. Compare with solutions generated via finite differences.

(b) Assorted applications using one-dimensional finite elements or spectral methods.

3. Use Taylor series to create numerical routines (stencils) to approximate the solutions of partial differential equations in two dimensions.

   Core Criteria:

   (a) Derive and implement a five-point finite difference stencil of to numerically approximate the solution of a second order boundary value problem of the form $\bigtriangledown^2 u(x, y) = F(x, y)$ with boundary conditions $u(a, y) = f(y)$, $u(b, y) = g(y)$, $u(x, c) = p(x)$ and $u(x, d) = r(x)$ on the rectangular spacial domain $[a, b] \times [c, d]$.

   (b) Derive and implement a finite difference stencil to numerically approximate the solution of a two-dimensional, rectangular geometry, initial-value, boundary-value, parabolic partial differential equation over a given spacial domain for a given time interval. Possible solution methods include the following:

      Two-dimensional-explicit (forward time) method.

      Two-dimensional Crank-Nicolson method.

      The Alternating Direction Method.

   (c) Derive and implement a two-dimensional wave equation numerical solution routine based on (explicit) centered finite differences to solve problems of the form:

      $u_{tt} = c^2(u_{xx} + u_{yy})$ for $0 \leq x \leq a$ and $0 \leq y \leq b$, with $t = [0, T]$. The boundary and initial conditions will be of the form $u(x, y, 0) = f(x, y), u_t(x, y, 0) = F(x, y) \quad u(x, y, t) = 0$, for all $x$ and $y$ on the (rectangular) boundary.

   Additional Criteria:

   (a) Derive and implement a two-dimensional wave equation (in polar coordinates) numerical solution routine based on (explicit) centered finite differences to solve problems of the form:

      $u_{tt} = c^2(u_{xx} + u_{yy})$ for $0 \leq r \leq a$ and $0 \leq \theta \leq b$, with $t = [0, T]$. The boundary and initial conditions will be of the form $u(r, \theta, 0) = f(r, \theta), u_t(r, \theta, 0) = F(r, \theta) \quad u(r, \theta, t) = 0$, for all $r$ and $\theta$ on the (circular) boundary. Part of this exercise is to transform the partial differential equation from cartesian to polar coordinates.

   (b) Derive and implement a two-dimensional wave equation numerical solution routine based on finite volumes to solve problems of the form:

      $u_{tt} = c^2(u_{xx} + u_{yy})$ for $0 \leq x \leq a$ and $0 \leq y \leq b$, with $t = [0, T]$. The boundary and initial conditions will be of the form $u(x, y, 0) = f(x, y), u_t(x, y, 0) = F(x, y) \quad u(x, y, t) =$

0, for all $x$ and $y$ on the (rectangular) boundary. Compare with solutions generated via finite difference methods.

(c) Assorted applications using two-dimensional finite elements or spectral methods.

(d) Assorted examples with adaptive mesh refinement.

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